

# CARATHÉODORY'S THEOREM AND MODULI OF LOCAL CONNECTIVITY

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ABSTRACT. We give a quantitative proof of the Carathéodory Theorem by means of the concept of a modulus of local connectivity and the extremal distance of the separating curves of an annulus.

## 1. INTRODUCTION

The goal of this paper is to give a new proof of the Carathéodory Theorem which states that if  $D$  is a Jordan domain, and if  $\phi$  is a conformal map of  $D$  onto the unit disk, then  $\phi$  extends to a homeomorphism of  $\overline{D}$  with the closed unit disk (see e.g. [4], [5], and [9]). This proof has a feature which appears to be new in that for each  $\zeta \in \partial D$  it explicitly constructs a  $\delta$  for each  $\epsilon$  when proving the existence of  $\lim_{z \rightarrow \zeta} \phi(z)$ . Furthermore, a closed form expression for  $\delta$  in terms of  $\epsilon$  and  $\zeta$  is obtained. Such expressions are potentially useful when estimating error in numerical computations. This is accomplished by means of a *modulus of local connectivity* for the boundary of  $D$ . Roughly speaking, this is a function that predicts how close two boundary points must be in order to connect them with a small arc that is included in the boundary. As in [9], the proof uses the extremal distance of the separating curves of an annulus to bound  $|\phi(z) - \phi(\zeta)|$ .

The paper is organized as follows. Section 2 covers background material. Section 3 states the main ideas of the proof. Sections 4 and 5 deal with topological preliminaries. Our estimates are proven in Section 6 and Section 7 completes the proof.

## 2. BACKGROUND

Let  $\mathbb{N}$  denote the set of non-negative integers.

When  $\mathcal{A}$  is an annulus with inner radius  $r$  and outer radius  $R$ , let

$$\lambda(\mathcal{A}) = \frac{2\pi}{\log(R/r)}.$$

$\lambda(\mathcal{A})$  is the extremal length of the family of separating curves of  $\mathcal{A}$ ; see e.g. [3]. Note that  $\lambda(\mathcal{A})$  decreases as the annulus  $\mathcal{A}$  gets thicker (i.e. as the ratio  $R/r$  increases) and increases as  $\mathcal{A}$  gets thinner (i.e. as the ratio  $R/r$  decreases).

When  $X$ ,  $Y$ , and  $Z$  are subsets of the plane, we say that  $X$  *separates*  $Y$  from  $Z$  if  $Y$  and  $Z$  are included in distinct connected components of  $\mathbb{C} - X$ . In the case where  $Y = \{p\}$ , we say that  $X$  separates  $p$  from  $Z$ . In the case where  $Y = \{p\}$  and  $Z = \{q\}$  we say that  $X$  separates  $p$  from  $q$ .

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A topological space is *locally connected* if it has a basis of open connected sets. By the Hahn-Mazurkiewicz Theorem, every curve is locally connected; see e.g. Section 3-5 of [6]. Suppose  $X$  is a compact and connected metric space. Then,  $X$  is locally connected if and only if it is *uniformly locally arcwise connected*. This means that for every  $\epsilon > 0$ , there is a  $\delta > 0$  so that whenever  $p, q \in X$  and  $0 < d(p, q) < \delta$ ,  $X$  includes an arc from  $p$  to  $q$  whose diameter is smaller than  $\epsilon$  (although its length may be infinite); again, see Section 3-5 of [6]. Accordingly, we define a *modulus of local connectivity* for a metric space  $X$  to be a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  so that whenever  $p, q \in X$  and  $0 < d(p, q) \leq 2^{-f(k)}$ ,  $X$  includes an arc from  $p$  to  $q$  whose diameter is smaller than  $2^{-k}$ . Thus, a metric space is uniformly locally arcwise connected if and only if it has a modulus of local connectivity, and a metric space that is compact and connected is locally connected if and only if it has a modulus of local connectivity. Note that if  $f$  is a modulus of local connectivity, then  $\lim_{k \rightarrow \infty} f(k) = \infty$ . In addition, if a metric space has a modulus of local connectivity, then it has a modulus of local connectivity that is increasing.

Moduli of local connectivity originated in the adaptation of local connectivity properties to the setting of theoretical computer science in [1] and [2]. Computational connections between moduli of local connectivity and boundary extensions of conformal maps are made in [7]. Here, we show that this notion may be useful in more traditional mathematical settings.

### 3. OUTLINE OF THE PROOF

We first observe the following which is proven in Section 4.

**Theorem 3.1.** *If  $\zeta_0$  is a boundary point of a simply connected Jordan domain  $D$ , then for every  $r > 0$ ,  $\zeta_0$  is a boundary point of exactly one connected component of  $D_r(\zeta_0) \cap D$ .*

Suppose  $\zeta_0$  is a boundary point of a simply connected Jordan domain  $D$ . In light of Theorem 3.1, when  $r > 0$  we let  $C(D; \zeta_0, r)$  denote the connected component of  $D_r(\zeta_0) \cap D$  whose boundary contains  $\zeta_0$ . Suppose  $\phi$  is a conformal map of  $D$  onto the unit disk. The fundamental strategy of the proof is to bound the diameter of  $\phi[C(D; \zeta_0, r)]$ . To do so, we first construct an upper bound on the diameter of  $\phi[C]$  where  $C$  is a connected component of  $D_r(\zeta) \cap D$  for some point  $\zeta$  in the complement of  $D$ . Namely, in Section 6 we prove the following.

**Theorem 3.2.** *Let  $\phi$  be a conformal map of a domain  $D$  onto the unit disk. Suppose  $\mathcal{A}$  is an annulus so that  $\overline{\mathcal{A}}$  separates its center from  $\phi^{-1}[\overline{D_r(0)}]$  where  $r \geq \sqrt{\pi\lambda(\mathcal{A})}$ . Let  $C$  be a connected component of the points of  $D$  that are inside the inner circle of  $\mathcal{A}$ . Suppose  $l = 1 - \sqrt{r^2 - \pi\lambda(\mathcal{A})}$ . Then, the diameter of  $\phi[C]$  is at most  $\sqrt{l^2 + 4\pi\lambda(\mathcal{A})}$ .*

Note that Theorem 3.2 applies to non-Jordan domains.

With Theorem 3.2 in hand, some basic calculations, which we perform in Section 6, lead us to the following.

**Theorem 3.3.** *Suppose  $\phi$  is a conformal map of a Jordan domain  $D$  onto the unit disk. Let  $\zeta_0$  be a boundary point of  $D$ , and let  $\epsilon > 0$ . Then, the diameter of  $\phi[C(D; \zeta_0, r_0)]$  is smaller than  $\epsilon$  whenever  $r_0$  is a positive number that is smaller*

than

$$(3.1) \quad \sup_{0 < l < \epsilon} \left( \exp \left( \frac{8\pi^2}{l^2 - \epsilon^2} \right) \min \left\{ |\zeta_0 - \phi^{-1}(w)| : |w| \leq \sqrt{(1-l)^2 + \frac{\epsilon^2 - l^2}{4}} \right\} \right).$$

When  $0 < \epsilon < 1$  and  $l = \frac{\epsilon}{2}$ ,

$$\frac{7}{16} < (1-l)^2 + \frac{\epsilon^2 - l^2}{4} < 1.$$

Thus, (3.1) is positive when  $0 < \epsilon < 1$ . In other words, for all sufficiently small  $\epsilon > 0$ , there is a positive number  $r_0$  that is smaller than (3.1).

So, suppose  $\phi$  is a conformal map of a Jordan domain  $D$  onto the unit disk. We use Theorem 3.3 to form an extension of  $\phi$  to  $\overline{D}$  as follows. Let  $\zeta_0$  be a boundary point of  $D$ . Note that  $C(D; \zeta_0, r') \subseteq C(D; \zeta_0, r)$  when  $0 < r' < r$ . It follows from Theorem 3.3 that there is exactly one point in

$$\bigcap_{r>0} \overline{\phi[C(D; \zeta_0, r)]}.$$

We define this point to be  $\phi(\zeta_0)$ .

Our next goal is to show that this extension of  $\phi$  is continuous. That is,  $\lim_{z \rightarrow \zeta} \phi(z) = \phi(\zeta)$  whenever  $\zeta$  is a boundary point of  $D$ . This is accomplished by showing that  $z \in C(D; \zeta, r)$  whenever  $z \in D$  is sufficiently close to  $\zeta$ . This is where we use moduli of local connectivity. Namely, in Section 4 we prove the following.

**Theorem 3.4.** *Suppose  $g$  is a modulus of local connectivity for a Jordan curve  $\sigma$ . Suppose  $D$  is an open disk whose boundary separates two points of  $\sigma$ . Suppose  $z_0$  and  $\zeta_0$  are points so that  $\zeta_0 \in \sigma \cap D$ ,  $z_0 \in D - \sigma$ , and  $|z_0 - \zeta_0| < 2^{-g(k)}$  where  $2^{-k} + 2^{-g(k)} \leq \max\{d(\zeta_0, \partial D), d(z_0, \partial D)\}$ . Then,  $\zeta_0$  is a boundary point of the connected component of  $z_0$  in  $D - \sigma$ .*

Theorem 3.4 was previously proven by means of the Carathéodory Theorem in [8]. We give another proof here with a few extra topological steps so as to avoid circular reasoning.

We then obtain the following form of the Carathéodory Theorem from Theorems 3.3 and Theorem 3.4.

**Theorem 3.5.** *Suppose  $\phi$  is a conformal map of a Jordan domain  $D$  onto the unit disk. Let  $\zeta_0$  be a boundary point of  $D$ . Then,  $\lim_{z \rightarrow \zeta_0} \phi(z) = \phi(\zeta_0)$ . Furthermore, if  $g$  is a modulus of local connectivity for the boundary of  $D$ , then for each  $\epsilon > 0$ ,  $|\phi(z_0) - \phi(\zeta_0)| < \epsilon$  whenever  $z_0$  is a point in  $D$  so that  $|z_0 - \zeta_0| < 2^{-g(k)}$  and  $k$  is a non-negative integer so that  $2^{-k} + 2^{-g(k)}$  is smaller than (3.1). Finally, the extension of  $\phi$  to  $\overline{D}$  is a homeomorphism of  $\overline{D}$  with the closed unit disk.*

The proof of Theorem 3.5 is given in Section 7.

Suppose  $\phi$ ,  $D$ ,  $g$ ,  $\zeta_0$  are as in Theorem 3.5. Without loss of generality suppose  $g$  is increasing. Thus  $2^{-k} + 2^{-g(k)} \leq 2^{-k+1}$ . Let  $0 < \epsilon < 1$ . We define a positive

number  $\delta(\zeta_0, \epsilon)$  so that  $|\phi(z) - \phi(\zeta_0)| < \epsilon$  when  $|z - \zeta_0| < \delta(\zeta_0, \epsilon)$ . Let:

$$\begin{aligned} k(\zeta_0, \epsilon) &= 2 - \left\lfloor \sup_{0 < l < \epsilon} \left( \frac{8\pi^2}{l^2 - \epsilon^2} + \right. \right. \\ &\quad \left. \left. \min \left\{ \log |\zeta_0 - \phi^{-1}(w)| : |w| \leq \sqrt{(1-l)^2 + \frac{\epsilon^2 - l^2}{4}} \right\} \right) \right\rfloor \\ \delta(\zeta_0, \epsilon) &= 2^{-k(\zeta_0, \epsilon)} + 2^{-g(k(\zeta_0, \epsilon))} \end{aligned}$$

(Here,  $\lfloor x \rfloor$  denotes the largest integer that is not larger than  $x$ .) Thus, by Theorem 3.5,  $|\phi(z) - \phi(\zeta_0)| < \epsilon$  whenever  $z \in D$  and  $|z - \zeta_0| < \delta(\zeta_0, \epsilon)$ .

#### 4. PROOFS OF THEOREMS 3.1 AND 3.4

Theorem 3.4 is used to prove Theorem 3.1. The proof of Theorem 3.4 is based on the following lemma and theorem.

**Lemma 4.1.** *Let  $D$  be a Jordan domain. Let  $\alpha$  be a crosscut of  $D$ , and let  $\gamma_1, \gamma_2$  be the subarcs of the boundary of  $D$  that join the endpoints of  $\alpha$ . Then, the interior of  $\gamma_1 \cup \alpha$  is one side of  $\alpha$ , and the interior of  $\gamma_2 \cup \alpha$  is the other side of  $\alpha$ .*

*Proof.* Let  $U_j$  denote the interior of  $\alpha \cup \gamma_j$ . Choose a point  $p$  in  $\alpha \cap D$ . There is a positive number  $\delta$  so that  $D_\delta(p) \subseteq D$ . Since  $p$  is a boundary point of  $U_j$ ,  $U_j \cap D_\delta(p)$  is non-empty. So, let  $q_j \in U_j \cap D_\delta(p)$ , and let  $D_j$  be the side of  $\alpha$  that contains  $q_j$ .

We show that  $U_j = D_j$ .  $U_j$  is a connected subset of  $D - \alpha$  that contains a point of  $D_j$  (namely  $q_j$ ). So,  $U_j \subseteq D_j$ . On the other hand,  $D_j$  is a connected subset of  $\mathbb{C} - (\gamma_j \cup \alpha)$  that contains a point of  $U_j$ . So,  $D_j \subseteq U_j$ .

$D_1 \neq D_2$  since  $\partial D_1 \neq \partial D_2$ . Thus,  $U_1$  and  $U_2$  are the two sides of  $\alpha$ .  $\square$

**Theorem 4.2.** *Let  $D$  be an open disk, and let  $\sigma$  be a Jordan curve. Suppose the boundary of  $D$  separates two points of  $\sigma$ . Let  $C$  be a connected component of  $D - \sigma$ . Then,  $C$  is the interior of a Jordan curve. Furthermore, if  $p$  is a boundary point of  $C$  that also lies in  $D$ , then  $p$  lies on  $\sigma$  and the boundary of  $C$  includes the connected component of  $p$  in  $D \cap \sigma$ .*

*Proof.* Since  $C \neq D$ , the boundary of  $C$  contains a point of  $\sigma$ ; let  $p$  denote such a point.

Since the boundary of  $D$  separates two points of  $\sigma$ , if  $G$  is a connected component of  $D \cap \sigma$ , then  $\overline{G}$  is a crosscut of  $D$ .

Let  $E$  denote the connected component of  $p$  in  $\sigma \cap D$ . Since  $C$  is a connected subset of  $D - E$ , there is a side of  $E$  that includes  $C$ ; let  $E^-$  denote this side, and let  $E^+$  denote the other side. By Lemma 4.1, each of these sides is a Jordan domain. Again, since the boundary of  $D$  separates two points of  $\sigma$ , if  $G$  is a connected component of  $\sigma \cap E^-$ , then  $\overline{G}$  is a crosscut of  $E^-$ .

We aim to show that the boundary of  $C$  is a Jordan curve which includes  $E$ . To this end, we construct an arc  $F$  so that  $E \cup F$  is a Jordan curve whose interior is  $C$ .  $F$  will be a union of subarcs of  $\sigma$  and connected subsets of the boundary of  $D$ . To define these subarcs of  $\sigma$ , we define a partial ordering of the connected components of  $\sigma \cap E^-$ . Namely, when  $G_1, G_2$  are connected components of  $\sigma \cap E^-$ , write  $G_1 \prec G_2$  if  $G_2$  is between  $G_1$  and  $E$ ; that is if  $E$  and  $G_1$  lie in opposite sides of  $\overline{G_2}$ .

Since  $\sigma$  is locally connected, it follows that there is no increasing chain  $G_1 \prec G_2 \prec G_3 \prec \dots$ . It then follows that if  $G_1$  is a connected component of  $\sigma \cap E^-$ , then there is a  $\preceq$ -maximal component of  $\sigma \cap E^-$ ,  $G$ , so that  $G_1 \preceq G$ .

We now define  $F$ . Let  $F' = \partial E^- \cap \partial D$ . Thus,  $E \cup F' = \partial E^-$ . Let  $\mathcal{M}$  denote the set of all  $\preceq$ -maximal components of  $\sigma \cap E^-$ . For each  $G \in \mathcal{M}$ , let  $\lambda_G$  be the subarc of  $F'$  that joins the endpoints of  $\overline{G}$ . Let  $F$  be formed by removing each  $\lambda_G$  from  $F'$  and replacing it with  $\overline{G}$ .

Thus,  $F$  is an arc that joins the endpoints of  $E$  and that contains no other points of  $E$ . Let  $J = E \cup F$ . Then,  $J$  is a Jordan curve. We show that  $C$  is the interior of  $J$ . Note that since  $J \subseteq \overline{E^-}$ ,  $E^-$  includes the interior of  $J$ .

When  $G \in \mathcal{M}$ , let  $G^+$  be the side of  $\overline{G}$  that includes  $E$  (when  $\overline{G}$  is viewed as a crosscut of  $D$  rather than  $E^-$ ), and let  $G^-$  denote the other side. The rest of the proof revolves around the following four claims.

- (1) For each  $G \in \mathcal{M}$ , the exterior of  $J$  includes  $G^-$ .
- (2) The interior of  $J$  includes  $\bigcap_{G \in \mathcal{M}} G^+ \cap E^-$ .
- (3) For each  $G \in \mathcal{M}$ ,  $G^+$  includes  $C$ .
- (4) The interior of  $J$  contains no point of  $\sigma$ .

Claims (2) and (3) together imply that the interior of  $J$  includes  $C$ . Claim (1) will be used to prove (4). Claim (4) shows that the interior of  $J$  is included in a connected component of  $D - \sigma$  which then must be  $C$ .

We begin by proving (1). Let  $p' \in G^-$ . Let  $z_0 \in \mathbb{C} - \overline{D}$ . Thus,  $z_0$  is exterior to  $J$  since  $J \subseteq \overline{D}$ . We construct an arc from  $p'$  to  $z_0$  that contains no point of  $J$ . Let  $q \in \lambda_G - \overline{G}$ . By Lemma 4.1,  $G^-$  is the interior of  $G \cup \lambda_G$ . So, there is an arc  $\sigma_1$  from  $p'$  to  $q$  so that  $\sigma'_1 \cap \partial G^- = \{q\}$ . There is an arc  $\sigma_2$  from  $q$  to  $z_0$  so that  $\sigma_2 \cap \partial D = \{q\}$ . Thus,  $\sigma_1 \cup \sigma_2$  is an arc from  $p'$  to  $z_0$  that contains no point of  $J$ . Thus,  $p'$  is exterior to  $J$  for every  $p' \in G^-$ .

We now prove (2). Suppose  $p_0 \in E^-$  belongs to  $G^+$  for every  $G \in \mathcal{M}$ . By way of contradiction, suppose  $p_0$  is exterior to  $J$ . Again, let  $z_0 \in \mathbb{C} - \overline{D}$ . Thus, the exterior of  $J$  includes an arc from  $p_0$  to  $z_0$ ; let  $\alpha$  denote such an arc. By examination of cases,  $\alpha$  cannot cross the boundary of  $D$  at any boundary point of  $E^-$ . So, it must do so at a boundary point of  $E^+$ . But, this entails that  $\alpha$  crosses  $E$  which it does not since  $J$  includes  $E$ . This is a contradiction, and so  $p_0$  is interior to  $J$ .

Next, we prove (3). Let  $G \in \mathcal{M}$ . Since  $\sigma$  is locally connected, and since  $p \in E$ , there is a positive number  $\delta$  so that  $D_\delta(p)$  contains no point of any connected component of  $\sigma \cap E^-$ . However, this disk must contain a point of  $C$ ,  $p'$ . So,  $[p', p]$  contains a point of  $E$  but no point of  $G$ . Hence,  $p' \in G^+$ . Since  $C$  is a connected subset of  $D - G$ ,  $C \subseteq G^+$ .

Finally, we prove (4). By way of contradiction, suppose  $p'$  is a point on  $\sigma$  that is interior to  $J$ . As noted above,  $E^-$  includes the interior of  $J$ . So,  $p' \in \sigma \cap E^-$ . Let  $G_1$  be the connected component of  $p'$  in  $\sigma \cap E^-$ . Let  $G$  be a  $\preceq$ -maximal component of  $\sigma \cap E^-$  so that  $G_1 \preceq G$ . Since  $p'$  lies inside  $J$ , and since  $J$  includes  $G$ ,  $p' \notin G$ . So,  $G_1 \prec G$ . This means that  $G_1 \subseteq G^-$ . By (1),  $p'$  is exterior to  $J$ —a contradiction. So, the interior of  $J$  contains no point of  $\sigma$ .

By the remarks after (4),  $C$  is the interior of  $J$  and the proof is complete.  $\square$

*Proof of Theorem 3.4.* Let  $C$  be the connected component of  $z_0$  in  $D - \sigma$ . Let  $l = [z_0, \zeta_0]$ . Let  $z_1$  be the point in  $l \cap \sigma$  that is closest to  $z_0$ . Thus,  $z_1 \in \partial C$ . Since

$|z_1 - \zeta_0| < 2^{-g(k)}$ ,  $\sigma$  includes an arc from  $z_1$  to  $\zeta_0$  whose diameter is smaller than  $2^{-k}$ ; call this arc  $\sigma_1$ .

We claim that  $D$  includes  $\sigma_1$ . For, let  $q \in \sigma_1$ . It follows that

$$\max\{|q - z_0|, |q - \zeta_0|\} < 2^{-k} + 2^{-g(k)}.$$

Since  $2^{-k} + 2^{-g(k)} \leq \max\{d(\zeta_0, \partial D), d(z_0, \partial D)\}$ , it follows that  $q \in D$ .

Since  $\sigma_1 \subseteq D$ ,  $\zeta_0$  belongs to the connected component of  $z_1$  in  $D \cap \sigma$ . By the ‘Furthermore’ part of Theorem 4.2, the boundary of  $C$  includes this component. Thus,  $\zeta_0$  is a boundary point of  $C$ .  $\square$

*Proof of Theorem 3.1.* Without loss of generality, suppose  $D_r(\zeta_0)$  does not include  $D$ . Let  $J$  denote the boundary of  $D$ . It follows that  $\partial D_r(\zeta_0)$  separates two points of  $J$ .

It follows from Theorem 3.4 that  $\zeta_0$  is a boundary point of at least one connected component of  $D_r(\zeta_0) - J$ . We now show it is a boundary point of exactly two such components. Let  $E$  be the connected component of  $\zeta_0$  in  $D_r(\zeta_0) \cap J$ . Thus, as noted in the proof of Theorem 4.2,  $\overline{E}$  is a crosscut of  $D_r(\zeta_0)$ . If  $C$  is a connected component of  $D_r(\zeta_0) - J$ , and if  $\zeta_0$  is a boundary point of  $C$ , then exactly one side of  $E$  includes  $C$ . By the proof of Theorem 3.1, if  $C$  is a connected component of  $D_r(\zeta_0) - J$ , then the side of  $E$  that includes  $C$  completely determines the boundary of  $C$ . Thus,  $\zeta_0$  is a boundary point of exactly two connected components of  $D - J$ ; one for each side of  $E$ .

So, let  $C_1, C_2$  denote the two connected components of  $D_r(\zeta_0) - J$  whose boundaries contain  $\zeta_0$ . Each of these components is a connected subset of  $\mathbb{C} - J$ . So each is either included in the interior of  $J$  or in the exterior of  $J$ . Since there are points of the interior and exterior of  $J$  that are arbitrarily close to  $\zeta_0$ , it follows from Theorem 3.4 that one of these components is included in the interior of  $J$  and one is included in the exterior of  $J$ . Suppose  $C_1$  is included in the interior of  $J$ ; that is,  $D \supseteq C_1$ .

Let  $p \in C_1$ , and let  $U$  be the connected component of  $p$  in  $D \cap D_r(\zeta_0)$ . We show that  $U = C_1$ . Since  $C_1$  is a connected subset of  $D \cap D_r(\zeta_0)$  that contains  $p$ ,  $C_1 \subseteq U$ . Since  $U$  is a connected subset of  $D_r(\zeta_0) - J$  that contains  $p$ ,  $U \subseteq C_1$ . This completes the proof of the theorem.  $\square$

## 5. PRELIMINARIES TO PROOF OF THEOREM 3.2: POLAR SEPARATIONS

**Definition 5.1.** Let  $\mathcal{A}$  be an annulus, and let  $\Omega$  be an open subset of  $\mathcal{A}$ . A *polar separation* of the boundary of  $\Omega$  is a pair of disjoint sets  $(E, F)$  so that whenever  $C$  is an intermediate circle of  $\mathcal{A}$ , there is a connected component of  $C \cap \Omega$  whose boundary contains a point of  $E$  and a point of  $F$ .

Our goal in this section is to prove the following.

**Theorem 5.2.** *Let  $\mathcal{A}$  be an annulus, and let  $D$  be a simply connected Jordan domain. Suppose that  $\mathcal{A}$  separates two boundary points of  $D$ , and let  $\gamma_1$  and  $\gamma_2$  be the subarcs of the boundary of  $D$  that join these points. Then,  $(\gamma_1 \cap \mathcal{A}, \gamma_2 \cap \mathcal{A})$  is a polar separation of the boundary of  $D \cap \mathcal{A}$ .*

Our proof of Theorem 5.2 is based on the following lemma.

**Lemma 5.3.** *Let  $C$  be a circle, and let  $D$  be a simply connected Jordan domain. Suppose  $C$  separates two boundary points of  $D$ . Then, there is a connected component of  $C \cap D$  whose boundary hits both subarcs of the boundary of  $D$  that join these two boundary points of  $D$ .*

*Proof.* Let  $p$  be a boundary point of  $D$  that is exterior to  $C$ , and let  $q$  be a boundary point of  $D$  that belongs to the interior of  $C$ .

Let  $\gamma_1, \gamma_2$  denote the subarcs of the boundary of  $D$  that join  $p$  and  $q$ . Let  $\alpha$  be a crosscut of  $D$  so that  $\alpha \cap C$  consists of a single point; label this point  $p'$ . Let  $D_j$  denote the interior of  $\alpha \cup \gamma_j$ . By Lemma 4.1,  $D_1$  and  $D_2$  are the sides of  $\alpha$ .

Now, for each  $j \in \{1, 2\}$ , we construct a point  $q_j$  in  $C \cap D_j$  so that  $p'$  is a boundary point of the connected component of  $q_j$  in  $C \cap D_j$ . Since  $D$  is open, there is a positive number  $\delta$  so that  $D_\delta(p') \subseteq D$ . Let  $C' = C \cap D_\delta(p')$ . Thus,  $C'$  is a subarc of  $C$ . Let  $q \in C' - \{p'\}$ . Then,  $q \notin \alpha$  since  $C \cap \alpha = \{p'\}$ . So,  $q \in D_1 \cup D_2$ . Without loss of generality, suppose  $q \in D_1$ . Relabel  $q$  as  $q_1$ . Let  $q_2$  be a point of  $C'$  so that  $p'$  is between  $q_1$  and  $q_2$  on  $C'$ . Again,  $q_2 \in D_1 \cup D_2$ . Since  $D_1$  is the interior of a Jordan curve, and since the subarc of  $C'$  from  $q_1$  to  $q_2$  crosses the boundary of  $D_1$  exactly once,  $q_2 \notin D_1$ . So,  $q_2 \in D_2$ .

Let  $E_j$  denote the connected component of  $q_j$  in  $C \cap D_j$ . By construction,  $p'$  is a boundary point of  $E_j$ . So, the other endpoint of  $E_j$  must be in  $\gamma_j$  since  $C \cap \alpha = \{p'\}$ . Set  $E = E_1 \cup E_2$ . Thus,  $E$  is a connected component of  $C \cap D$ . One endpoint of  $E$  belongs to  $\gamma_1$ , and the other belongs to  $\gamma_2$ . This proves the lemma.  $\square$

*Proof of Theorem 5.2.* By assumption,  $\mathcal{A}$  separates two boundary points of  $D$ . One of these points is interior to the inner circle of  $\mathcal{A}$ , and the other is exterior to the outer circle of  $\mathcal{A}$ . Let  $p$  denote a point that is exterior to the outer circle of  $\mathcal{A}$ , and let  $q$  denote a point that is interior to the inner circle of  $\mathcal{A}$ .

Let  $C$  be an intermediate circle of  $\mathcal{A}$ . Then,  $p$  is exterior to  $C$  and  $q$  is interior to  $C$ . So, by Lemma 5.3, there is a connected component of  $C \cap D$  so that one of its endpoints lies on  $\gamma_1$  and the other lies on  $\gamma_2$ . Thus,  $(\gamma_1 \cap \mathcal{A}, \gamma_2 \cap \mathcal{A})$  is a polar separation of the boundary of  $D \cap \mathcal{A}$ .  $\square$

## 6. PROOF OF THEOREMS 3.2 AND 3.3

When  $X, Y \subseteq \mathbb{C}$ , let  $d_{\inf}(X, Y)$  denote the infimum of  $|z - w|$  as  $z$  ranges over all points of  $X$  and  $w$  ranges over all points of  $Y$ .

The proof of the following is essentially the same as the proof of Lemma 4.1 of [7] which is a standard length-area argument.

**Lemma 6.1.** *Let  $\mathcal{A}$  be an annulus, and let  $\Omega$  be an open subset of  $\mathcal{A}$ . Suppose  $(E, F)$  is a polar separation of the boundary of  $\Omega$ . Then,*

$$\lambda(\mathcal{A}) \geq \sup_{\phi} \frac{d_{\inf}(\phi[E], \phi[F])^2}{\text{Area}(\phi[\Omega])}$$

where  $\phi$  ranges over all maps that are conformal on a neighborhood of  $\overline{\Omega}$ .

*Proof of Theorem 3.2.* Note that  $r < 1$  since  $C$  is non-empty.

We begin by constructing a rectangle  $R$  as follows. Let  $z_0$  be any point of  $\phi[C]$ . Choose  $m, l_0$  so that  $l_0 > l$ ,  $m > \sqrt{\pi\lambda(\mathcal{A})}$ , and  $(1 - l_0)^2 + m^2 < (1 - l)^2 + \pi\lambda(\mathcal{A})$ . Since  $r^2 = (1 - l)^2 + \pi\lambda(\mathcal{A})$ ,  $z$  is exterior to the outer circle of  $\mathcal{A}$  whenever  $|\phi(z)| \leq$

$\sqrt{(1-l_0)^2 + m^2}$ . Let:

$$\begin{aligned}\nu_1 &= \frac{z_0}{|z_0|}(1-l_0+mi) \\ \nu_2 &= \frac{z_0}{|z_0|}(1-l_0-mi)\end{aligned}$$

Thus, the radius  $[0, z_0/|z_0|]$  is a perpendicular bisector of the line segment  $[\nu_1, \nu_2]$ . The midpoint of  $[\nu_1, \nu_2]$  is  $(1-l_0)z_0/|z_0|$ , and the length of  $[\nu_1, \nu_2]$  is  $2m$ . Let:

$$\begin{aligned}\nu_3 &= \frac{z_0}{|z_0|}(1+mi) \\ \nu_4 &= \frac{z_0}{|z_0|}(1-mi)\end{aligned}$$

Thus, the line segment  $[\nu_3, \nu_4]$  is perpendicular to the radius  $[0, z_0/|z_0|]$ . Furthermore, the length of this segment is  $2m$  and its midpoint is  $z_0/|z_0|$ .

Let  $R$  be the open rectangle whose vertices are  $\nu_1, \nu_2, \nu_3$ , and  $\nu_4$ . That is,  $R$  is the interior of  $[\nu_1, \nu_3] \cup [\nu_3, \nu_4] \cup [\nu_4, \nu_2] \cup [\nu_2, \nu_1]$ .

Note that the diameter of  $R$  is  $\sqrt{l_0^2 + 4m^2}$ . Also, the diameter of  $R$  approaches  $\sqrt{l^2 + 4\pi\lambda(\mathcal{A})}$  as  $(l_0, m) \rightarrow (l, \sqrt{\pi\lambda(\mathcal{A})})$ . It thus suffices to show that  $\phi[C] \subseteq R$ .

We claim that it suffices to show that  $\phi[C]$  contains no boundary point of  $R$ . For, since  $\phi^{-1}(z_0)$  is interior to the outer circle of  $\mathcal{A}$ , the modulus of  $z_0$  is larger than  $\sqrt{(1-l_0)^2 + m^2}$  which is larger than  $1-l_0$ . This implies that  $z_0 \in R$ . Since  $R$  contains at least one point of  $\phi[C]$ , namely  $z_0$ , and since  $\phi[C]$  is connected, it suffices to show that  $\phi[C]$  contains no boundary point of  $R$ .

Since  $[\nu_3, \nu_4]$  contains no point of the unit disk, it contains no point of  $\phi[C]$ . By construction,  $|\nu_1| = |\nu_2| = \sqrt{(1-l_0)^2 + m^2}$ . Thus,  $|z| \leq \sqrt{(1-l_0)^2 + m^2}$  whenever  $z \in [\nu_1, \nu_2]$ . It follows from what has been observed about  $l_0$  and  $m$  that  $[\nu_1, \nu_2]$  contains no point of  $\phi[C]$ . So, it suffices to show that  $[\nu_1, \nu_3] \cup [\nu_4, \nu_2]$  contains no point of  $\phi[C]$ .

Let us begin by showing that  $[\nu_1, \nu_3]$  contains no point of  $\phi[C]$ . By way of contradiction, suppose otherwise. In order to obtain a contradiction, we construct a Jordan curve  $J$  so that  $\mathcal{A}$  separates two points of  $J$  as follows. Let  $z_1$  be a point of  $\phi[C]$  that belongs to  $[\nu_1, \nu_3]$ . Thus, by what has just been observed,  $z_1 \neq \nu_1$ . Let  $\sigma_0$  be the pre-image of  $\phi$  on  $[\nu_1, 0]$ . Let  $\sigma'_1$  be the pre-image of  $\phi$  on  $[\nu_1, z_1]$ . Let  $\sigma'_3$  be the pre-image of  $\phi$  on  $[0, z_0]$ . Since  $C$  is connected, it includes an arc from  $\phi^{-1}(z_1)$  to  $\phi^{-1}(z_0)$ ; label this arc  $\sigma'_2$ . Let  $w_1$  be the first point on  $\sigma'_1$  that belongs to  $\sigma'_2$ . Let  $w_2$  be the first point on  $\sigma'_3$  that belongs to  $\sigma'_2$ . Let  $\sigma_1$  be the subarc of  $\sigma'_1$  from  $\phi^{-1}(\nu_1)$  to  $w_1$ , and let  $\sigma_3$  be the subarc of  $\sigma'_3$  from  $w_2$  to  $\phi^{-1}(0)$ . Let  $\sigma_2$  be the subarc of  $\sigma'_2$  from  $w_1$  to  $w_2$ . Let  $J = \sigma_0 \cup \sigma_1 \cup \sigma_2 \cup \sigma_3$ . Thus,  $J$  is a Jordan curve. By construction,  $\mathcal{A}$  separates two points of  $J$ .

Let  $D'$  denote the interior of  $J$ . Let  $\Omega = D' \cap \mathcal{A}$ . Let  $E = \sigma_1 \cap \mathcal{A}$ , and let  $F = \sigma_3 \cap \mathcal{A}$ . We claim that  $(E, F)$  is a polar separation of the boundary of  $\Omega$ . For, let  $p = \phi^{-1}(\nu_1)$ , and let  $q = w_1$  (where  $w_1$  is as in the construction of  $J$ ). Thus,  $p$  is exterior to the outer circle of  $\mathcal{A}$ . Since  $q \in C$ ,  $q$  is interior to the inner circle of  $\mathcal{A}$ . Let  $\gamma_1 = \sigma_1$ , and let  $\gamma_2 = \sigma_2 \cup \sigma_3 \cup \sigma_0$ . Therefore,  $\gamma_1, \gamma_2$  are the subarcs of the boundary of  $D'$  that join  $p$  and  $q$ . So, by Theorem 5.2,  $(\gamma_1 \cap \mathcal{A}, \gamma_2 \cap \mathcal{A})$  is a polar separation of the boundary of  $\Omega$ . Since  $\sigma_0$  is the pre-image of  $\phi$  on  $[\nu_1, 0]$ ,  $\sigma_0$



contains no point of  $\overline{\mathcal{A}}$ . Since  $\sigma_2 \subseteq C$ ,  $\sigma_2$  contains no point of  $\overline{\mathcal{A}}$ . Thus,  $E = \gamma_1 \cap \mathcal{A}$ , and  $F = \gamma_2 \cap \mathcal{A}$ . Hence,  $(E, F)$  is a polar separation of the boundary of  $\Omega$ .

By construction,  $d_{\inf}(\phi[E], \phi[F]) = m$ . So, by Lemma 6.1, the area of  $\phi[\Omega]$  is at least as large as

$$m^2 \lambda(\mathcal{A})^{-1} > \pi.$$

This is impossible since the unit disk includes  $\phi[\Omega]$ . Thus,  $[\nu_1, \nu_3]$  contains no point of  $\phi[C]$ .

By similar reasoning,  $[\nu_4, \nu_2]$  contains no point of  $\phi[C]$ . Thus,  $\phi[C] \subseteq R$ , and the theorem is proven.  $\square$

*Proof of Theorem 3.3.* Suppose  $r_0$  is a positive number that is smaller than (3.1). We begin by defining an annulus  $\mathcal{A}$  as follows. Choose  $l$  so that  $0 < l < \epsilon$  and so that

$$r_0 < \exp\left(\frac{8\pi^2}{l^2 - \epsilon^2}\right) \min\left\{|\zeta_0 - \phi^{-1}(w)| : |w| \leq \sqrt{(1-l)^2 + \frac{\epsilon^2 - l^2}{4}}\right\}.$$

There is a positive number  $r_1$  so that

$$r_1 < \min\left\{|\zeta_0 - \phi^{-1}(w)| : |w| \leq \sqrt{(1-l)^2 + \frac{1}{4}(\epsilon^2 - l^2)}\right\}$$

and so that

$$r_0 < \exp\left(\frac{8\pi^2}{l^2 - \epsilon^2}\right) r_1.$$

Since  $l < \epsilon$ ,  $r_0 < r_1$ . So, define  $\mathcal{A}$  to be the annulus whose center is  $\zeta_0$ , whose outer radius is  $r_1$ , and whose inner radius is  $r_0$ .

We now show that the diameter of  $\phi[C(D; \zeta_0, r_0)]$  is smaller than  $\epsilon$ . First, note that  $\pi\lambda(\mathcal{A}) < (\epsilon^2 - l^2)/4$ . Set  $r = \sqrt{(l-1)^2 + \pi\lambda(\mathcal{A})}$ . It follows that  $|\zeta_0 - z| > r_1$  whenever  $|\phi(z)| \leq r$ . For, if  $|\phi(z)| \leq r$ , then  $|\phi(z)| < \sqrt{(l-1)^2 + (\epsilon^2 - l^2)/4}$  and so  $r_1 < |\zeta_0 - z|$  by the choice of  $r_1$ . This means that  $\mathcal{A}$  separates its center from  $\phi^{-1}[\overline{D_r(0)}]$ . By Theorem 3.2, the diameter of  $\phi[C(D; \zeta_0, r_0)]$  is at most

$$\sqrt{l^2 + 4\pi\lambda(\mathcal{A})}.$$

We have

$$\begin{aligned} l^2 + 4\pi\lambda(\mathcal{A}) &= l^2 + \frac{8\pi^2}{\log(r_1/r_0)} \\ &< l^2 + \epsilon^2 - l^2 = \epsilon^2. \end{aligned}$$

Thus, the diameter of  $\phi[C(D; \zeta_0, r_0)]$  is smaller than  $\epsilon$ .  $\square$

## 7. PROOF OF THE CARATHÉODORY THEOREM

We now conclude with the proof of Theorem 3.5. Set  $r_0 = 2^{-k} + 2^{-g(k)}$ . By Theorem 3.4,  $z_0 \in C(D; \zeta_0, r_0)$ . By Theorem 3.3,  $|\phi(z_0) - \phi(\zeta_0)| < \epsilon$ . Thus,  $\lim_{z \rightarrow \zeta_0} \phi(z) = \phi(\zeta_0)$ .

We now show that this extension of  $\phi$  is injective. It suffices to show that  $\phi(\zeta_0) \neq \phi(\zeta_1)$  whenever  $\zeta_0$  and  $\zeta_1$  are distinct boundary points of  $D$ . By way of contradiction, suppose  $\phi(\zeta_0) = \phi(\zeta_1)$ . Let  $p = \phi(\zeta_0)$ .

We construct a Jordan curve  $\sigma$  as follows. Let  $\alpha$  be a crosscut of  $D$  that joins  $\zeta_0$  and  $\zeta_1$ . Thus,  $\phi[\alpha]$  is a Jordan curve that contains no unimodular point other than  $p$ . Let  $\sigma = \phi[\alpha]$ .

We now construct an annulus  $\mathcal{A}$  that separates two points of  $\sigma$ . Fix a positive number  $R$  so that  $R < \max\{|z - p| : z \in \sigma\}$ . Choose another positive number  $r$  so that  $r < R$ . Let  $\mathcal{A}$  be the annulus whose center is  $p$ , whose inner radius is  $r$ , and whose outer radius is  $R$ . By the choice of  $R$ , there is a point  $q \in \sigma$  that is exterior to the outer circle of  $\mathcal{A}$ . Let  $\gamma_1$  and  $\gamma_2$  be the subarcs of  $\sigma$  that join  $p$  and  $q$ . Let  $E = \gamma_1 \cap \mathcal{A}$ , and let  $F = \gamma_2 \cap \mathcal{A}$ . Finally, let  $\Omega = \mathcal{A} \cap \mathbb{D}$  (where  $\mathbb{D}$  is the unit disk). Then, by Theorem 5.2,  $(E, F)$  is a polar separation of the boundary of  $\Omega$ . Now, since  $R$  is fixed, as  $r \rightarrow 0^+$ ,  $\lambda(\mathcal{A}) \rightarrow 0$ . However, by the choice of  $R$ ,  $d_{\inf}(E, F)$  is bounded away from 0 as  $r \rightarrow 0^+$ . Thus, by Lemma 6.1 (applied to  $\phi^{-1}$ ),  $\text{Area}(\phi^{-1}[\Omega]) \rightarrow \infty$  as  $r \rightarrow 0^+$ . Since  $\phi^{-1}[\Omega] \subseteq D$ , this is a contradiction. Thus,  $\phi(\zeta_0) \neq \phi(\zeta_1)$ .

Finally, we show that this extension of  $\phi$  is surjective. Let  $\zeta$  be a point on the unit circle. It follows from the Balzano-Weierstrauss Theorem that there is a boundary point of  $D$ ,  $\zeta_1$ , so that  $\zeta_1 \in \overline{\{\phi^{-1}(r\zeta) : 0 < r < 1\}}$ . Thus,  $\phi(\zeta_1) = \zeta$  by the continuity of  $\phi$ .

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#### REFERENCES

- [1] P.J. Couch, B.D. Daniel, and T.H. McNicholl, *Computing space-filling curves*, Theory of Computing Systems **50** (2012), no. 2, 370–386.
- [2] D. Daniel and T.H. McNicholl, *Effective local connectivity properties*, Theory of Computing Systems **50** (2012), no. 4, 621 – 640.
- [3] J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2005.
- [4] G. M. Golusin, *Geometric theory of functions of a complex variable*, American Mathematical Society, 1969.
- [5] R. Greene and S. Krantz, *Function theory of one complex variable*, Graduate Studies in Mathematics, American Mathematical Society, 2002.
- [6] John G. Hocking and Gail S. Young, *Topology*, second ed., Dover Publications Inc., New York, 1988.
- [7] T.H. McNicholl, *Computing boundary extensions of conformal maps*, To appear in London Mathematical Society Journal of Computational Mathematics.
- [8] ———, *Computing links and accessing arcs*, Mathematical Logic Quarterly **59** (2013), no. 1 - 2, 101 – 107.
- [9] Bruce P. Palka, *An introduction to complex function theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1991.

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